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Rational Interpolation to e^x

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1. INTRODUCTION

We derive estimates for the error in interpolating e^x by rational functions of degree n on intervals of length less than two. Let π_n denote the class of all polynomials of degree at most n with real coefficients. Our main result is the following:

THEOREM 1. *Let $\gamma_1, \gamma_2, ..., \gamma_{2n+1}$ be points (not necessarily distinct) in $[0, \alpha]$, where $\alpha < 2$. Choose $P_n, Q_n \in \pi_n$ so that*

$$P_n(\gamma_i) - Q_n(\gamma_i)\, e^{-\gamma_i} = 0 \qquad \text{for} \quad i = 1, 2, ..., 2n+1.$$

Then, for $x \in [0, \alpha]$,

$$|P_n(x)/Q_n(x) - e^{-x}| \leqslant \left(\frac{2e\sqrt{n}\, e^{2\sqrt{\alpha n}}}{2-\alpha}\right) \frac{n!(n+1)!}{(2n)!(2n+1)!} \left| \prod_{i=1}^{2n+1} (x - \gamma_i) \right| .$$

Furthermore, Q_n has positive coefficients.

Let

$$\lambda_{m,n}[a, b] = \min_{p \in \pi_m, q \in \pi_n} \|e^x - p_m(x)/q_n(x)\|_{[a,b]},$$

where $\|\cdot\|$ denotes the supremum norm on $[a, b]$.

The following conjecture was made by G. Meinardus in 1964.

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CONJECTURE [3, p. 168].

$$\lambda_{m,n}[-1,1] = \frac{m!\,n!}{2^{m+n}(m+n)!(m+n+1)!}(1+o(1)).$$

D. J. Newman, through some clever manipulation of the Padé approximant, has recently proved

THEOREM A [5, p. 24].

$$\lambda_{m,n}[-1,1] \leqslant \frac{8m!\,n!}{2^{m+n}(m+n)!(m+n+1)!}.$$

G. Németh ([4], see also Braess [1]) has shown

THEOREM B.

$$\lambda_{n,n}[-1,1] = \frac{n!\,n!}{4^n(2n)!(2n+1)!}(1+O(1)).$$

If we choose the γ_i in Theorem 1 to be the zeros of the $(2n+1)$st Chebyshev polynomial (shifted to $[0,\alpha]$) then we see that, up to the "slowly growing" $e^{2\sqrt{\alpha n}}$ term, we get essentially the right order of approximation. In light of Theorems A and B it seems plausible that the initial bracketed term of the error estimate is superfluous.

2. PRELIMINARIES

Suppose that $P_n, Q_n \in \pi_n$ and suppose that $P_n(x) - Q_n(x)\,e^{-x}$ has $2n+1$ zeros on the interval $[0,\alpha]$. If $Q_n(x) = q_0 + q_1 x + \cdots + q_n x^n$ then

$$\begin{aligned}
&(P_n(x) - Q_n(x)\,e^{-x})^{(n+1)} \\
&\quad = (Q_n(x)\,e^{-x})^{(n+1)} \\
&\quad = \sum_{k=0}^{n} \binom{n+1}{k} Q_n^{(k)} e^{-x}(-1)^{(n+1-k)} \\
&\quad = (-1)^{n+1}\,e^{-x} \sum_{k=0}^{n} \frac{x^k}{k!} \sum_{j=0}^{n-k} \binom{n+1}{j} (-1)^j (k+j)!\,q_{k+j}.
\end{aligned} \tag{1}$$

Since $(Q_n(x)\,e^{-x})^{(n+1)}$ has n zeros on $[0,\alpha]$, we deduce that there exist $\beta_1,\dots,\beta_n \in [0,\alpha]$ so that

$$\sum_{k=0}^{n} \frac{x^k}{k!} \sum_{j=0}^{n-k} \binom{n+1}{j} (-1)^j (k+j)!\,q_{k+j} = q_n \prod_{i=1}^{n} (x-\beta_i).$$

Thus, if $q_n \prod_{i=1}^{n}(x-\beta_i) = b_0 + b_1 x + \cdots + b_n x^n$, we have

$$\begin{bmatrix} \binom{n+1}{0}, & -\binom{n+1}{1}, & +\binom{n+1}{2}, & \ldots, & (-1)^n \binom{n+1}{n} \\ 0, & \binom{n+1}{0}, & -\binom{n+1}{1}, & \ldots, & (-1)^{n-1}\binom{n+1}{n-1} \\ 0, & 0, & \binom{n+1}{0}, & \ldots, & (-1)^{n-2}\binom{n+1}{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0, & 0, & 0, & \ldots, & \binom{n+1}{0} \end{bmatrix} \begin{bmatrix} q_0 0! \\ q_1 1! \\ q_2 2! \\ \vdots \\ q_n n! \end{bmatrix} = \begin{bmatrix} b_0 0! \\ b_1 1! \\ b_2 2! \\ \vdots \\ b_n n! \end{bmatrix}. \tag{2}$$

We can invert (2) to obtain

$$\begin{bmatrix} \binom{n}{n}, & \binom{n+1}{n}, & \binom{n+2}{n}, & \ldots, & \binom{2n}{n} \\ 0, & \binom{n}{n}, & \binom{n+1}{n}, & \ldots, & \binom{2n-1}{n} \\ 0, & 0, & \binom{n}{n}, & \ldots, & \binom{2n-2}{n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0, & 0, & 0, & \ldots, & \binom{n}{n} \end{bmatrix} \begin{bmatrix} b_0 0! \\ b_1 1! \\ b_2 2! \\ \vdots \\ b_n n! \end{bmatrix} = \begin{bmatrix} q_0 0! \\ q_1 1! \\ q_2 2! \\ \vdots \\ q_n n! \end{bmatrix}. \tag{3}$$

We observe that (3) can be easily derived from (2) combined with the facts that the (m, n) Padé approximant to e^{-x} is given by

$$\sum_{v=0}^{m} \frac{\binom{m}{v}}{\binom{m+n}{v}} \frac{(-x)^v}{v!} \Bigg/ \sum_{v=0}^{n} \frac{\binom{n}{v}}{\binom{n+m}{v}} \frac{x^v}{v!}.$$

and that for the Padé approximant $b_0 = b_1 = \cdots = b_{n-1} = 0$.

We are now in a position to prove the following:

LEMMA 1. *Suppose that* $P_n(x) \in \pi_n$ *and suppose that* $Q_n = q_0 +$

$q_1x + \cdots + q_nx^n$, where $q_0 > 0$. Suppose also that $P_n(x) - Q_n(x)e^{-x}$ has $2n+1$ zeros at $\gamma_1,\ldots,\gamma_{2n+1} \in [0,\alpha]$. Then, if $\alpha < 2$, Q_n has positive coefficients and

$$q_n \leqslant \left(\frac{2}{2-\alpha}\right)\frac{n!}{(2n)!}q_0.$$

Proof. The first part follows from an examination of (3) using the facts that for $i \leqslant n$,

$$(i-1)!\,|b_{i-1}| \leqslant \alpha(i!)\,|b_i| \qquad \text{and} \qquad \binom{n+i-1}{n} \leqslant \frac{1}{2}\binom{n+i}{n}.$$

The second part is proved by noting that

$$q_0 \geqslant n!\,|b_n|\binom{2n}{n} - (n-1)!\,|b_{n-1}|\binom{2n-1}{n}$$
$$\geqslant \left(1-\frac{\alpha}{2}\right)\frac{(2n)!}{n!}q_n. \quad \blacksquare$$

The next lemma is a slight adaptation of a result of S. N. Bernstein [2, p. 38].

LEMMA 2. *Suppose that f and g are $m+1$ times continuously differentiable on $[a,b]$ and suppose that $f(x) = g(x) = 0$ has $m+1$ solutions on $[a,b]$. If*

$$|f^{(m+1)}(x)| \leqslant g^{(m+1)}(x) \qquad \textit{for} \quad x \in [a,b]$$

then

$$|f(x)| \leqslant |g(x)| \qquad \textit{for} \quad x \in [a,b].$$

LEMMA 3 [3, pp. 16 and 165]. (a) *If $\gamma_1,\ldots,\gamma_{m+n+1} \in [a,b]$ then there exist $P_m \in \pi_m$, $Q_n \in \pi_n$, so that*

$$P_m(\gamma_i) - Q_n(\gamma_i)e^{-\gamma_i} = 0 \qquad \textit{for} \quad i = 1,2,\ldots,n+m+1.$$

(b) *If $P_m^* \in \pi_m$, $Q_n^* \in \pi_n$ and*

$$\|e^{-x} - P_m^*/Q_n^*\|_{[a,b]} = \min_{P_m \in \pi_m, Q_n \in \pi_n} \|e^{-x} - P_m/Q_n\|_{[a,b]}$$

then P_m^*/Q_n^* interpolates e^{-x} at exactly $n+m+1$ points in $[a,b]$.

3. Proof of Theorem 1

Lemma 3 guarantees the existence of P_n and Q_n with the desired interpolation property. We may assume that

$$Q_n(x) = q_0 + \cdots + q_{n-1}x^{n-1} + x^n.$$

Then, as in (1), there exist $\beta_1,..., \beta_n \in [0, \alpha]$ so that

$$\begin{aligned}(Q_n(x)\, e^{-x})^{(n+1)} &= (-1)^{n+1}\, e^{-x} \prod_{i=1}^{n} (x-\beta_i)\\ &= (-1)^{n+1}\, e^{-x} R_n(x).\end{aligned}$$

Hence,

$$(Q_n(x)\, e^{-x})^{(2n+1)} = (-1)^{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k\, e^{-x} R_n^{(n-k)}(x).$$

Since $R_n^{(n-k)}(x) = n!/k! \prod_{i=1}^{k}(x - \rho_{i,k})$, where $\rho_{i,k} \in [0, \alpha]$, we have

$$\begin{aligned}|(Q_n(x)\, e^{-x})^{(2n+1)}| &\leqslant \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} \alpha^k\\ &\leqslant n! \sum_{k=0}^{n} \frac{n!\, \alpha^k}{k!\, k!(n-k)!}.\end{aligned}$$

By Stirling's formula, $n^n e^{-n} < n! < e\sqrt{n}\, n^n e^{-n}$,

$$\begin{aligned}\frac{n!\, \alpha^k}{k!\, k!(n-k)!} &\leqslant \frac{e\sqrt{n}\, \alpha^k e^k n^n}{k^k k^k (n-k)^{n-k}}\\ &= e\sqrt{n}\, \alpha^k e^k \frac{n^k}{k^{2k}} \left(1 + \frac{k}{n-k}\right)^{n-k}\\ &\leqslant e\sqrt{n} \left(\frac{\alpha e^2 n}{k^2}\right)^k.\end{aligned}$$

A little elementary calculus reveals that $(\alpha e^2 n/k^2)^k$ has a maximum at $k = \sqrt{\alpha n}$ and hence,

$$|(Q_n(x)\, e^{-x})^{(2n+1)}| \leqslant (n+1)!\, e\sqrt{n}\, e^{2\sqrt{\alpha n}}.$$

We apply Lemma 2 using $m = 2n+1$,

$$f(x) = P_n(x) - Q_n(x)\, e^{-x},$$

and

$$g(x) = e\sqrt{n}\, e^{2\sqrt{\alpha n}} \frac{(n+1)!}{(2n+1)!} \prod_{i=1}^{2n+1} (x - \gamma_i)$$

and deduce that for $x \in [0, \alpha]$,

$$|P_n(x) - Q_n(x)\, e^{-x}| \leqslant e\sqrt{n}\, e^{2\sqrt{\alpha n}} \frac{(n+1)!}{(2n+1)!} \left| \prod_{i=1}^{2n+1} (x - \gamma_i) \right| .$$

We complete the result by appealing to Lemma 1 to show that for $x \geqslant 0$,

$$Q_n(x) \geqslant q_0 \geqslant \frac{(2-\alpha)}{2} \frac{(2n)!}{n!}. \quad \blacksquare$$

The (1, 1) Padé approximant to e^{-x} has denominator $Q(x) = 1 + \frac{1}{2}x$. It follows that the (1, 1) rational function that interpolates e^{-x} with multiplicity three at any point β will have denominator $Q_\beta(x) = 1 + \frac{1}{2}(x - \beta)$. In particular if $\beta \geqslant 2$ then Q_β does not have positive coefficients. This shows that $\alpha < 2$ is essential, at least for the $n = 1$ case, in Theorem 1.

References

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